Primitive Leavitt path algebras, and a general solution to a question of Kaplansky

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Noncommutative rings and their Applications, III

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(joint work with Jason Bell and K.M. Rangaswamy)

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Assume R at least has "local units":

Prime rings

Definition: I, J two-sided ideals of R. The product IJ is the two-sided ideal

$$IJ = \{\sum_{\ell=1}^{n} i_\ell j_\ell \mid i_\ell \in I, j_\ell \in J, n \in \mathbb{N}\}.$$

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Examples:

1. Commutative domains, e.g. fields, $\mathbb{Z}$, $K[x]$, $K[x, x^{-1}]$, ...
2. Simple rings
3. $\mathrm{End}_K(V)$ where $\dim_K(V)$ is infinite. $(\cong \mathrm{RFM}(K))$

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Lemma: R prime. Then R embeds as an ideal in a unital prime ring R_1. (Dorroh extension of R.)

If R is a K-algebra then we can construct R_1 a K-algebra for which $\dim_K(R_1/R) = 1$.

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Examples:

- Simple rings (note: need local units to build irreducibles)

NON-Examples:

- $\mathbb{Z}$, $K[x]$, $K[x, x^{-1}]$

Primitive rings

Primitive rings are "natural" generalizations of matrix rings.

Jacobson's Density Theorem: R is primitive if and only if R is isomorphic to a dense subring of $\mathrm{End}_D(V)$, for some division ring D, and some D-vector space V.

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Definition of "primitive" makes sense for non-unital rings.

Prime and primitive rings

Well-known (and easy) **Proposition**: Every primitive ring is prime.

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If R is prime, then in previous embedding,

$$R \text{ is primitive} \Leftrightarrow R_1 \text{ is primitive.}$$

Prime and primitive rings

Converse of Lemma is not true (e.g. $\mathbb{Z}$, $K[x]$, $K[x, x^{-1}]$).

In fact, the only commutative primitive unital rings are fields.

Leavitt path algebras

Let $E = (E^0, E^1, r, s)$ be any directed graph, and K any field.

$$\bullet^{s(e)} \xrightarrow{e} \bullet^{r(e)}$$

Construct the "double graph" (or "extended graph") $\widehat{E}$, and then the path algebra $K\widehat{E}$.

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(CK1) $e^*e = r(e)$; $f^*e = 0$ for $f \neq e$ in E^1; and

(CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for all $v \in E^0$

(just at those vertices v which are not *sinks*)

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Then the *Leavitt path algebra of E with coefficients in K* is:

$$L_K(E) = K\widehat{E} \ / \ < (CK1), (CK2) >$$

Leavitt path algebras: Examples

Example 1.

$$E = \bullet^{v_1} \xrightarrow{e_1} \bullet^{v_2} \xrightarrow{e_2} \bullet^{v_3} \cdots\cdots\cdots \bullet^{v_{n-1}} \xrightarrow{e_{n-1}} \bullet^{v_n}$$

Then $L_K(E) \cong \mathrm{M}_n(K)$.

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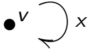
Example 3.

$$E = \bullet^{v_1} \xrightarrow{(\mathbb{N})} \bullet^{v_2}$$

Then $L_K(E) \cong \mathrm{FM}_{\mathbb{N}}(K)_1$.

Example 4.

$$E = R_1 =$$



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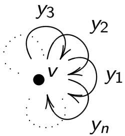

Then $L_K(E) \cong L_K(1, n)$, the *Leavitt algebra of type* $(1, n)$.

Leavitt path algebras: basic properties

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2. Every element of $L_K(E)$ can be expressed as $\sum_{i=1}^{n} k_i \alpha_i \beta_i^*$ where $k_i \in K$ and α_i, β_i are paths for which $r(\alpha_i) = r(\beta_i)$. (This is not generally a basis.)

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3. There is a natural $\mathbb{Z}$-grading on $L_K(E)$, generated by defining

$$\deg(v) = 0, \quad \deg(e) = 1, \quad \deg(e^*) = -1$$

With respect to this grading, every nonzero graded ideal of $L_K(E)$ contains a vertex of E.

Leavitt path algebras: basic properties

4. An *exit* e for a cycle $c = e_1 e_2 \cdots e_n$ based at v is an edge for which $s(e) = s(e_i)$ for some $1 \leq i \leq n$, but $e \neq e_i$.

If every cycle in E has an exit ("Condition (L)"), then every nonzero ideal of $L_K(E)$ contains a vertex, and every nonzero left ideal of $L_K(E)$ contains a nonzero idempotent.

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5. If c is a cycle based at v for which c has no exit, then $vL_K(E)v \cong K[x, x^{-1}]$.

Prime Leavitt path algebras

Notation: $u \geq v$ means either $u = v$ or there exists a path p for which $s(p) = u, r(p) = v$. u "connects to" v.

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Lemma. If I is a two-sided ideal of $L_K(E)$, and $u \in E^0$ has $u \in I$, and $u \geq v$, then $v \in I$.

Easy proof: If p has $s(p) = u, r(p) = w$, then using (CK1) we get

$$p^*p = r(p) = w; \quad \text{but} \quad p^*p = p^* \cdot s(p) \cdot p = p^*up \in I.$$

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Theorem. (Aranda Pino, Pardo, Siles Molina 2009) E arbitrary. Then $L_K(E)$ is prime $\Leftrightarrow$ for each pair $v, w \in E^0$ there exists $u \in E^0$ with $v \geq u$ and $w \geq u$.

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Idea of Proof. $(\Rightarrow)$ Let R denote $L_K(E)$. Let $v, w \in E^0$. But $RvR \neq \{0\}$ and $RwR \neq \{0\} \Rightarrow RvRwR \neq \{0\} \Rightarrow vRw \neq \{0\} \Rightarrow$ $v\alpha\beta^* w \neq 0$ for some paths α, β in E. Then $u = r(\alpha)$ works.

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$(\Leftarrow)$ $L_K(E)$ is graded by $\mathbb{Z}$, so need only check primeness on nonzero graded ideals I, J. But each nonzero graded ideal contains a vertex. Let $v \in I \cap E^0$ and $w \in J \cap E^0$. By downward directedness there is $u \in E^0$ with $v \geq u$ and $w \geq u$. But then $u = p^* vp \in I$ and $u = q^* wq \in J$, so that $0 \neq u = u^2 \in IJ$.

The Countable Separation Property

Definition. Let E be any directed graph. E has the *Countable Separation Property* (CSP) if there exists a countable set of vertices S in E for which every vertex of E connects to an element of S.

E has the "Countable Separation Property" with respect to S.

The Countable Separation Property

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Example: X uncountable, S the set of finite subsets of X. Define the graph E_X:

1. vertices indexed by S, and
2. edges induced by proper subset relationship.

Then E_X does not have CSP.

Primitive Leavitt path algebras

Can we describe the (left) primitive Leavitt path algebras?

Note: Since $L_K(E) \cong L_K(E)^{op}$, left primitivity and right primitivity coincide. So we can just say "primitive" Leavitt path algebra.

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2. every cycle in E has an exit (Condition (L)), and
3. E has the Countable Separation Property.

$L_K(E)$ primitive $\Leftrightarrow$ E has (MT3), (L), and CSP

Strategy of Proof:

1. (Easy) A unital ring R is left primitive if and only if there is a left ideal $N \neq R$ of R such that for every nonzero two-sided ideal I of R, $N + I = R$.

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3. Show that CSP allows us to build a left ideal in $L_K(E)_1$ with the desired properties.

4. Then show that the lack of the CSP implies that no such left ideal can exist in $L_K(E)_1$.

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($\Leftarrow$). Suppose E downward directed, E has Condition (L), and E has CSP.

Suffices to establish primitivity of $L_K(E)_1$. Let T denote a set of vertices w/resp. to which E has CSP.

T is countable: label the elements $T = \{v_1, v_2, ...\}$.

$L_K(E)$ primitive $\Leftarrow$ E has (MT3), (L), and CSP

Inductively define a sequence $\lambda_1, \lambda_2, \ldots$ of paths in E for which, for each $i \in \mathbb{N}$,

1. λ_i is an initial subpath of λ_j whenever $i \leq j$, and
2. $v_i \geq r(\lambda_i)$.

Define $\lambda_1 = v_1$.

Suppose $\lambda_1, \ldots, \lambda_n$ have the indicated properties. By downward directedness, there is u_{n+1} in E^0 for which $r(\lambda_n) \geq u_{n+1}$ and $v_{n+1} \geq u_{n+1}$. Let $p_{n+1} : r(\lambda_n) \rightsquigarrow u_{n+1}$.

Define $\lambda_{n+1} = \lambda_n p_{n+1}$. $\square$

$L_K(E)$ primitive $\Leftarrow$ E has (MT3), (L), and CSP

Since λ_i is an initial subpath of λ_t for all $i \leq t$, we get that

$$\lambda_i\lambda_i^*\lambda_t\lambda_t^* = \lambda_t\lambda_t^* \quad \text{for each pair of positive integers} \quad i \leq t.$$

In particular $(1 - \lambda_i\lambda_i^*)\lambda_t\lambda_t^* = 0$ for $i \leq t$.

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In particular $(1 - \lambda_i \lambda_i^*)\lambda_t \lambda_t^* = 0$ for $i \leq t$.

Define $N = \sum_{i=1}^{\infty} L_K(E)_1 (1 - \lambda_i \lambda_i^*)$.

$N \neq L_K(E)_1$: otherwise, $1 = \sum_{i=1}^{t} r_i(1 - \lambda_i \lambda_i^*)$ for some $r_i \in L_K(E)_1$, but then

$$0 \neq 1 \cdot \lambda_t \lambda_t^* = (\sum_{i=1}^{t} r_i(1 - \lambda_i \lambda_i^*)) \cdot \lambda_t \lambda_t^* = 0.$$

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Claim: Every nonzero two-sided ideal I of $L_K(E)_1$ contains some $\lambda_n \lambda_n^*$.

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Idea: E is downward directed, so $L_K(E)$, and therefore $L_K(E)_1$, is prime. Since $L_K(E)$ embeds in $L_K(E)_1$ as a two-sided ideal, we get $I \cap L_K(E)$ is a nonzero two-sided ideal of $L_K(E)$. So Condition (L) gives that I contains some vertex w.

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Then $w \geq v_n$ for some n by CSP. But $v_n \geq r(\lambda_n)$ by construction, so $w \geq r(\lambda_n)$. So $w \in I$ gives $r(\lambda_n) \in I$, so $\lambda_n \lambda_n^* \in I$.

Now we're done. Show $N + I = L_K(E)_1$ for every nonzero two-sided ideal I of $L_K(E)_1$. But $1 - \lambda_n \lambda_n^* \in N$ (all $n \in \mathbb{N}$) and $\lambda_n \lambda_n^* \in I$ (some $n \in \mathbb{N}$) gives $1 \in N + I$.

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For the converse:

1) If E is not downward directed then $L_K(E)$ not prime, so that $L_K(E)$ not primitive.

2) General ring theory result: If R is primitive and $f = f^2$ is nonzero then fRf is primitive.

So if E contains a cycle c (based at v) without exit then $vL_K(E)v \cong K[x, x^{-1}]$, which is not primitive, and thus $L_K(E)$ is not primitive.

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3) (The hard part.) Show if E does not have CSP then $L_K(E)$ is not primitive.

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Lemma. Let N be a left ideal of a unital ring A. If there exist $x, y \in A$ such that $1 + x \in N$, $1 + y \in N$, and $xy = 0$, then $N = A$.

Proof: Since $1 + y \in N$ then $x(1 + y) = x + xy = x \in N$, so that

$$1 = (1 + x) - x \in N.$$

$L_K(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

We show that if E does not have CSP, then there does NOT exist a left ideal $N \neq L_K(E)_1$ for which $N + I = L_K(E)_1$ for all two-sided ideals I of $L_K(E)_1$.

To do this: assume N is such an ideal, show $N = L_K(E)_1$.

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To do this: assume N is such an ideal, show $N = L_K(E)_1$.

Strategy: If N has this property, then for each $v \in E^0$ we have $N + \langle v \rangle = L_K(E)_1$. So for each $v \in E^0$ there exists $y_v \in \langle v \rangle$, $n_v \in N$ for which $n_v + y_v = 1$. Let $x_v = -y_v$. This gives a set $\{x_v \mid v \in E^0\} \subseteq L_K(E)_1$ for which $1 + x_v \in N$ for all $v \in E^0$.

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Now show that the lack of CSP in E^0 forces the existence of a pair of vertices v, w for which $x_v \cdot x_w = 0$. (This is the technical part.)

Then use the Lemma.

$L_K(E)$ primitive $\Rightarrow$ E has (MT3), (L), and CSP

Key pieces of the technical part:

1. Every element ℓ of $L_K(E)$ can be written as $\sum_{i=1}^{n} k_i \alpha_i \beta_i^*$ for some $n = n(\ell)$, and paths α_i, β_i. In particular, we can "cover" all elements of $L_K(E)$ by specifying n and lengths of paths. This is a countable covering of $L_K(E)$. (Not a partition.)

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Key pieces of the technical part:

1. Every element ℓ of $L_K(E)$ can be written as $\sum_{i=1}^{n} k_i \alpha_i \beta_i^*$ for some $n = n(\ell)$, and paths α_i, β_i. In particular, we can "cover" all elements of $L_K(E)$ by specifying n and lengths of paths. This is a countable covering of $L_K(E)$. (Not a partition.)
2. Collect up the x_v according to this covering. Since E does not have CSP, then some specific subset in the cover does not have CSP.

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2. Collect up the x_v according to this covering. Since E does not have CSP, then some specific subset in the cover does not have CSP.
3. Show that, in this specific subset Z, there exists $v \in Z$ for which the set
$$\{w \in Z \mid x_v x_w = 0\}$$
does not have CSP. In particular, this set is nonempty. Pick such v and w. Then we are done by the Lemma. □

von Neumann regular rings

Definition: R is *von Neumann regular* (or just *regular*) in case

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(R is not required to be unital.)

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Examples:

1. Division rings
2. Direct sums of matrix rings over division rings
3. Direct limits of von Neumann regular rings

R is regular $\Leftrightarrow$ R_1 is regular.

Kaplansky's Question

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Answered in the negative (Domanov, 1977), a group-algebra example. (Clever, but very ad hoc.)

Kaplansky's Question

Theorem. (A-, K.M. Rangaswamy 2010)

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Idea of Proof: $(\Leftarrow)$ If E contains a cycle c based at v, can show that $a = v + c$ has no "regular inverse".

$(\Rightarrow)$ Show that if E is acyclic then every element of $L_K(E)$ can be trapped in a subring of $L_K(E)$ which is isomorphic to a finite direct sum of finite matrix rings.

Application to Kaplansky's question

It's not hard to find acyclic graphs E for which $L_K(E)$ is prime but for which C.S.P. fails.

Example (mentioned previously): X uncountable, S the set of finite subsets of X. Define the graph E_X:

- vertices indexed by S, and

- edges induced by proper subset relationship.

Then for the graph E_X,

1. $L_K(E_X)$ is regular (E is acyclic)
2. $L_K(E_X)$ is prime (E is downward directed)
3. $L_K(E_X)$ is not primitive (E does not have CSP).

Application to Kaplansky's question

By using uncountable sets of different cardinalities, we get:

Theorem: For any field K, there exists an infinite class (up to isomorphism) of K-algebras (of the form $L_K(E_X)$) which are von Neumann regular and prime, but not primitive.

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Remark: These examples are also "Cohn path algebras".

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For these graphs E, embedding $L_K(E)$ in $L_K(E)_1$ in the usual way gives unital, regular, prime, not primitive algebras. So we get

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Theorem: For any field K, there exists an infinite class (up to isomorphism) of unital K-algebras (of the form $L_K(E_X)_1$) which are von Neumann regular and prime, but not primitive.

Remark: The algebras $L_K(E_X)_1$ are never Leavitt path algebras.

Application to Kaplansky's question

A different construction of germane graphs:

Let $\kappa > 0$ be any ordinal. Define E_κ as follows:

$$E_\kappa^0 = \{\alpha \mid \alpha < \kappa\}, \quad E_\kappa^1 = \{e_{\alpha,\beta} \mid \alpha, \beta < \kappa, \text{ and } \alpha < \beta\},$$

$s(e_{\alpha,\beta}) = \alpha$, and $r(e_{\alpha,\beta}) = \beta$ for each $e_{\alpha,\beta} \in E_\kappa^1$.

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Suppose κ has *uncountable cofinality*. Then E_κ is downward directed, and has Condition (L), but does not have CSP. This gives:

Theorem: If $\{\kappa_i \mid i \in I\}$ is a set of ordinals having distinct cardinalities, for which each κ_i has uncountable cofinality, then the set $\{L_K(E_{\kappa_i}) \mid i \in I\}$ is a set of nonisomorphic K-algebras, each of which is von Neumann regular, and prime, but not primitive.

Primitive graph C^*-algebras

An intriguing connection:

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Theorem. (A-, Mark Tomforde, in preparation)

Let E be any graph. Then $C^*(E)$ is primitive if and only if

1. E is downward directed,
2. E satisfies Condition (L), and
3. E satisfies the Countable Separation Property.

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Proofs of the sufficiency direction for $L_{\mathbb{C}}(E)$ and $C^*(E)$ results are dramatically different.

Questions?